ON BASIC RELATIONS IN THE THEORY OF THIN SHELLS

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Considered is the question of possible simplification of the so-called relations of elasticity in the theory of thin shells. It was thought that the simplest version of such relations (often referred to as the first approximation of Love's theory) could lead only to such an error which is of the order of $O(h/R)$ (h and R are the thickness and the smallest of the principal radii of curvature of the shell). In this connection an opinion was expressed [1] that the simplest elasticity relations for shear forces and twisting moments can lead to a more significant error. In the helical shell example of [2] there are shown the essential defects of the solution obtained on the basis of the simplest elasticity relations*. These defects, apparently, are connected with the simplification of the elasticity relations for shear forces and twisting moments. The simplest elasticity relations for normal forces and bending moments were not subjected to criticism; it appeared they should not contribute significantly to errors.

The present work establishes that this is not quite so. Initially it is shown how to obtain complete relations of elasticity on the basis of the widely known results of Love, and there is derived a supplementary (to the sixth's equilibrium equation) algebraic equation relating the shear forces with the twisting moments. Then it is determined which simplifications of the complete elasticity relations are permissible for a cylindrical shell. It is established with the example of such a shell that the inclusion of the usually ignored quantities in the elasticity relations for normal forces and bending moments can essentially affect the solution of some problems.

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* Reference [2] came to the author's attention after the present work was completed for publication.

The investigation carried out leads to the elasticity relations for an arbitrary shell which are basically not unlike those given in [3].

1. Let us refer to the linear theory of thin shells with constant thickness as presented by Love [4]. We will retain part of the notation used in [4] and will change the remaining part for convenience. In the first and third row below are shown notations used by Love, while in the second and fourth row are given the corresponding notations of this paper:

$2h$;	α, β;	A, B;	u, v, w;	$\tilde{\omega}$, τ;	e_{xx}, e_{yy}, e_{xy};	σ
h;	α_1, α_2;	A_1, A_2;	u_1, u_2, u_3;	ω, τ_*;	e_1, e_2, e_{12};	ν

X_x, Y_y, X_y;	S_1,	S_2,	G_i,	H_1,	H_2;	X',	Y',	Z',	L',	M'
σ_1, σ_2, σ_{12};	T_{12},	$-T_{21}$,	M_i,	$-M_{12}$,	M_{21};	P_1,	P_2,	P_3,	$\widetilde{M}_1$,	$\widetilde{M}_2$

Correct (within the limits of linear theory) values of the components of deformation at any point in the shell are determined by Formulas (30) derived in Chapter 24 of [4]. The first hypothesis of Kirchhoff-Love should be formulated as follows: in the three formulas for e_1, e_2, e_{12} mentioned above, one can neglect ξ, η, ζ, i.e. in defining e_1, e_2, e_{12} one may assume that the section of the normal to the middle surface of the shell constrained between its outer surfaces remains straight in the process of deformation, is normal to the middle surface and remains constant in length. On the basis of this hypothesis, we have

$$e_i = \frac{\varepsilon_i - z\varkappa_i}{1 - z / R_i} \tag{1.1}$$

$$e_{12} = \frac{\omega}{1 - z / R_2} - \tau_* z\left(\frac{1}{1 - z / R_1} + \frac{1}{1 + z / R_2}\right) + \frac{z}{1 - z / R_2}\left(\frac{p_1'}{A_1} + \frac{q_2'}{A_2}\right)$$

where $i = 1, 2$ as in the following. In the last formula (see [4], Chapt. 24, Equations (11))

$$\frac{p_1'}{A_1} + \frac{q_2'}{A_2} = \omega\frac{q_1'}{A_1} + \varepsilon_1\frac{p_1'}{A_1} + \varepsilon_2\frac{q_2'}{A_2}$$

Hence, neglecting quantities of higher order than the deformation of the middle surface (see [4], Chapt. 24, Equations (21), (26)) we obtain

$$\frac{p_1'}{A_1} + \frac{q_2'}{A_2} = -\frac{\omega}{R_1}$$

and consequently

$$e_{12} = \omega\frac{1 - z / R_1}{1 - z / R_2} - \tau_* z\left(\frac{1}{1 - z / R_1} + \frac{1}{1 - z / R_2}\right) \tag{1.2}$$

Quantities ϵ_i, κ_i, ω, τ_* are determined from the displacement components u_1, u_2, u_3 in the form (see [4], Chapt. 24, Equations (21), (26))

$$\varepsilon_i = \frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i} + \frac{u_{3-i}}{A_i A_{3-i}}\frac{\partial A_i}{\partial \alpha_{3-i}} - \frac{u_3}{R_i} \tag{1.3}$$

$$\varkappa_i = \frac{1}{A_i}\frac{\partial}{\partial \alpha_i}\left(\frac{1}{A_i}\frac{\partial u_3}{\partial \alpha_i} + \frac{u_i}{R_i}\right) + \frac{1}{A_i A_{3-i}}\left(\frac{1}{A_{3-i}}\frac{\partial u_3}{\partial \alpha_{3-i}} + \frac{u_{3-i}}{R_{3-i}}\right)\frac{\partial A_i}{\partial \alpha_{3-i}} \tag{1.4}$$

$$\omega = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) \tag{1.5}$$

$$\tau_* = \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} + \frac{u_2}{R_2}\right) - \frac{1}{A_1^2 A_2}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_1} - \frac{1}{A_1 R_1}\frac{\partial u_2}{\partial \alpha_1} \tag{1.6}$$

Let us introduce the quantity $\tau^* = \tau_* + \omega/R_1$. From the equalities (1.5), (1.6) and one of Codazzi's formulas

$$\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2}\right) = \frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1}$$

it is easy to establish that

$$\tau^* = \frac{1}{A_1 A_2}\frac{\partial^2 u_3}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_1 A_2^2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_2} - \frac{1}{A_2 A_1^2}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_1} + \frac{A_2}{A_1 R_2}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2 R_1}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)$$

By means of simple transformations it is possible now to represent the right-hand side of Formula (1.2) in a clearly invariant form relative to the transposition of notations of the coordinate lines

$$e_{12} = \omega\left(1 + \frac{z/R_1}{1 - z/R_1} + \frac{z/R_2}{1 - z/R_2}\right) - \tau^* z\left(\frac{1}{1 - z/R_1} + \frac{1}{1 - z/R_2}\right)$$

or in a more convenient and also clearly invariant form for the following

$$e_{12} = \omega_1 \frac{1 - z/R_2}{1 - z/R_1} + \omega_2 \frac{1 - z/R_1}{1 - z/R_2} - \tau z\left(\frac{1}{1 - z/R_1} + \frac{1}{1 - z/R_2}\right) \tag{1.7}$$

where

$$\omega_1 = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right), \qquad \omega_2 = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)$$
$$\tau = \frac{1}{A_1 A_2}\left(\frac{\partial^2 u_3}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_1}\right) \tag{1.8}$$

and where $\omega = \omega_1 + \omega_2$, as follows from Formula (5).

On the basis of the second Kirchhoff-Love hypothesis, stating that at any point in the shell the stress σ_3 is considered negligible* compared to the larger of the σ_1 or σ_2 stresses, and in view of Hooke's law we have

$$\sigma_i = \frac{E}{1-\nu^2}(e_i + \nu e_{3-i}), \qquad \sigma_{12} = \frac{E}{2(1+\nu)} e_{12} \tag{1.9}$$

Substituting e_i and e_{12} into (1.9) from Formulas (1.1) and (1.7) we obtain

$$\begin{gathered}\sigma_i = \frac{E}{1-\nu^2}\left(\frac{\varepsilon_i - z\varkappa_i}{1 - z/R_i} + \nu\frac{\varepsilon_{3-i} - z\varkappa_{3-i}}{1 - z/R_{3-i}}\right)\\ \sigma_{12} = \frac{E}{2(1+\nu)}\left[\omega_1\frac{1 - z/R_2}{1 - z/R_1} + \omega_2\frac{1 - z/R_1}{1 - z/R_2} - \tau z\left(\frac{1}{1 - z/R_1} + \frac{1}{1 - z/R_2}\right)\right]\end{gathered} \tag{1.10}$$

The forces and moments per unit length are determined from the equalities

$$\begin{gathered}T_i = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \sigma_i\left(1 - \frac{z}{R_{3-i}}\right)dz, \qquad T_{i,3-i} = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \sigma_{i,3-i}\left(1 - \frac{z}{R_{3-i}}\right)dz\\ M_i = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \sigma_i\left(1 - \frac{z}{R_{3-i}}\right)zdz, \qquad M_{i,3-i} = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \sigma_{i,3-i}\left(1 - \frac{z}{R_{3-i}}\right)zdz\end{gathered} \tag{1.11}$$

Substituting here in place of σ_i and $\sigma_{i,3-i}$ the right-hand sides of the equalities (1.10), we obtain

$$\begin{gathered}T_i = \frac{Eh}{1-\nu^2}\left[\varepsilon_i + \nu\varepsilon_{3-i} - \frac{\gamma_i}{12\beta_i}(\chi_i - \chi_{3-i})(\chi_i\varepsilon_i - h\varkappa_i)\right]\\ T_{i,3-i} = \frac{Eh}{2(1+\nu)}\left\{\omega + \frac{\gamma_i}{12\beta_i}(\chi_i - \chi_{3-i})\left[\tau h - \omega_i(\chi_i - \chi_{3-i})\right]\right\}\\ M_i = -\frac{Eh^2}{12(1-\nu^2)}\left[h(\varkappa_i + \nu\varkappa_{3-i}) + \frac{\gamma_i}{\beta_i}(\chi_i - \chi_{3-i})\varepsilon_i - \right.\\ \left. - h\left(1 - \frac{\chi_{3-i}}{\chi_i}\right)\left(1 + \frac{\gamma_i}{\beta_i}\right)\varkappa_i\right]\\ M_{i,3-i} = -\frac{Eh^2}{24(1+\nu)}\left\{\left[2 - \left(1 - \frac{\chi_{3-i}}{\chi_i}\right)\left(1 + \frac{\gamma_i}{\beta_i}\right)\right][h\tau - (\chi_i - \chi_{3-i})\omega_i] + \chi_i\omega\right\}\end{gathered} \tag{1.12}$$

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* It is understood, that we are excluding the case when the shell is loaded on both outer surfaces by equally distributed normal forces, which referred to the unit of the middle surface, have equal magnitude but opposite directions.

where

$$\chi_i = \frac{h}{R_i}, \qquad \beta_i = \frac{h^2}{12R_i^2}, \qquad \gamma_i = 1 - \chi_i^{-1} \ln \frac{1 + {}^1/_2\chi_i}{1 - {}^1/_2\chi_i}$$

whereby if $R_i = \infty$ ($\chi_i = 0$), then $\gamma_i/\beta_i = -1$ since

$$\gamma_i = -\beta_i - \frac{9}{5}\beta_i^2 - \ldots \tag{1.13}$$

and consequently

$$\frac{\gamma_i}{\beta_i} = -1 + O(\beta_i), \qquad 1 + \gamma_i/\beta_i = -\frac{9}{5}\beta_i(1 + O(\beta_i))$$

and then Formulas (1.12) can be expressed in the form

$$T_i = \frac{Eh}{1-\nu^2}\Big[\varepsilon_i + \nu\varepsilon_{3-i} + \frac{1}{12}(\chi_i - \chi_{3-i})(1 + O(\beta_i))(\chi_i\varepsilon_i - h\varkappa_i)\Big]$$

$$T_{i,3-i} = \frac{Eh}{2(1+\nu)}\Big\{\omega - \frac{1}{12}(\chi_i - \chi_{3-i})(1 + O(\beta_i))[\tau h - \omega_i(\chi_i - \chi_{3-i})]\Big\}$$

$$M_i = -\frac{Eh^2}{12(1-\nu^2)}\Big[h(\varkappa_i + \nu\varkappa_{3-i}) - (\chi_i - \chi_{3-i})(1 + O(\beta_i))\varepsilon_i + \frac{9}{5}h\beta_i(1 + O(\beta_i))\Big(1 - \frac{\chi_{3-i}}{\chi_i}\Big)\varkappa_i\Big] \tag{1.14}$$

$$M_{i,3-i} = -\frac{Eh^2}{24(1+\nu)}\Big\{\Big[2 + \frac{9}{5}\beta_i(1 + O(\beta_i)) \times \Big(1 - \frac{\chi_{3-i}}{\chi_i}\Big)\Big][h\tau - (\chi_i - \chi_{3-i})\omega_i] + \chi_i\omega\Big\}$$

In Formulas (1.12) for $T_{i,3-i}$, $M_{i,3-i}$ appears the expression $h\tau - (\chi_i - \chi_{3-i})\omega_i$ equal to $h\tau^* - \chi_i\omega$. Therefore, the quantities $T_{i,3-i}$, $M_{i,3-i}$ are linear functions of the two quantities ω and τ^*.

The forces and moments per unit length are related by the conditions of equilibrium of the shell (see [4], Chapt. 24, Equations (45) and (46))

$$\frac{\partial}{\partial\alpha_i}A_{3-i}T_i + \frac{\partial}{\partial\alpha_{3-i}}A_iT_{3-i,i} + T_{i,3-i}\frac{\partial A_i}{\partial\alpha_{3-i}} - T_{3-i}\frac{\partial A_{3-i}}{\partial\alpha_i} - A_iA_{3-i}\Big(\frac{N_i}{R_i} - P_i\Big) = 0 \tag{1.15}$$

$$\frac{\partial}{\partial\alpha_1}A_2N_1 + \frac{\partial}{\partial\alpha_2}A_1N_2 + A_1A_2\Big(\frac{T_1}{R_1} + \frac{T_2}{R_2} + P_3\Big) = 0 \tag{1.16}$$

$$\frac{\partial}{\partial\alpha_i}A_{3-i}M_i + \frac{\partial}{\partial\alpha_{3-i}}A_iM_{3-i,i} + M_{i,3-i}\frac{\partial A_i}{\partial\alpha_{3-i}} - M_{3-i}\frac{\partial A_{3-i}}{\partial\alpha_i} - A_iA_{3-i}(N_i - \widetilde{M}_i) = 0 \tag{1.17}$$

$$\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}-T_{12}+T_{21}=0 \tag{1.18}$$

The fulfilment of equality (1.18) is guaranteed by Formulas (1.11). If one substitutes into (1.18) the quantities T_{12}, T_{21}, M_{12}, M_{21} from (1.11) then, as is known, it will become an identity (the same will apparently take place by using (1.12)). The second algebraic equality relating T_{12}, T_{21}, M_{12}, M_{21} must follow from the fact that these four quantities are linear functions of the two quantities ω and τ^*.

Two independent algebraic equalities relating T_{12}, T_{21}, M_{12}, M_{21} can be obtained as follows. We will replace in Formulas (1.12) for T_{12} and T_{21} the quantity ω by the sum $\omega_1+\omega_2$, and we will consider them as a system of equations relative to ω_1 and ω_2 (it is easy to see that its determinant is positive for $R_1 \neq R_2$). Solving for ω_1 and ω_2 from the system, we can substitute these expressions into Formulas (1.12) for M_{12} and M_{21}. Thereby the terms containing τ are eliminated and the two desired equalities are obtained:

$$\left[1-\frac{\gamma_1}{\beta_1}+\frac{\chi_2}{\chi_1}\left(1+\frac{\gamma_1}{\beta_1}\right)-\frac{\gamma_1}{12\beta_1}\chi_1(\chi_1-\chi_2)\right]T_{21}-\left\{1-\frac{\gamma_1}{\beta_1}+\frac{\chi_2}{\chi_1}\left(1+\frac{\gamma_1}{\beta_1}\right)+\right.$$
$$\left.+\frac{\gamma_2}{12\beta_2}\frac{\chi_1-\chi_2}{\chi_1}\left[\chi_2^2+\frac{\gamma_1}{\beta_1}(\chi_1-\chi_2)^2\right]\right\}T_{12}=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left[\frac{\gamma_1}{\beta_1}+\frac{\gamma_2}{\beta_2}-\frac{\gamma_1\gamma_2}{12\beta_1\beta_2}(\chi_1-\chi_2)^2\right]M_{12} \tag{1.19}$$

$$\left[1-\frac{\gamma_2}{\beta_2}+\frac{\chi_1}{\chi_2}\left(1+\frac{\gamma_2}{\beta_2}\right)-\frac{\gamma_2}{12\beta_2}\chi_2(\chi_2-\chi_1)\right]T_{12}-\left\{1-\frac{\gamma_2}{\beta_2}+\frac{\chi_1}{\chi_2}\left(1+\frac{\gamma_2}{\beta_2}\right)+\right.$$
$$\left.+\frac{\gamma_1}{12\beta_1}\frac{\chi_2-\chi_1}{\chi_2}\left[\chi_1^2+\frac{\gamma_2}{\beta_2}(\chi_2-\chi_1)^2\right]\right\}T_{21}=\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left[\frac{\gamma_1}{\beta_1}+\frac{\gamma_2}{\beta_2}-\frac{\gamma_1\gamma_2}{12\beta_1\beta_2}(\chi_2-\chi_1)^2\right]M_{21} \tag{1.20}$$

These coincide only in the case when $R_1 = R_2$.

Multiplying (1.19) by χ_2, (1.20) by χ_1, adding the results and dividing out $\chi_2-\chi_1$, one obtains (1.18). Adding (1.19) and (1.20) we obtain, after reduction by $\chi_2-\chi_1$

$$\frac{h}{12}(\chi_1-\chi_2)\left[\frac{\gamma_1\gamma_2}{\beta_1\beta_2}(T_{12}+T_{21})-\frac{\chi_1\gamma_1}{\chi_2\beta_1}\left(1+\frac{\gamma_2}{\beta_2}\right)T_{21}-\frac{\chi_2\gamma_2}{\chi_1\beta_2}\left(1+\frac{\gamma_1}{\beta_1}\right)T_{12}\right]=$$
$$=\left[\frac{\gamma_1}{\beta_1}+\frac{\gamma_2}{\beta_2}-\frac{\gamma_1\gamma_2}{12\beta_1\beta_2}(\chi_1-\chi_2)^2\right](M_{21}-M_{12})-h\left[\frac{1}{\chi_1}\left(1+\frac{\gamma_1}{\beta_1}\right)+\frac{1}{\chi_2}\left(1+\frac{\gamma_2}{\beta_2}\right)\right](T_{21}-T_{12}) \tag{1.21}$$

This equality is fulfilled also when $R_1 = R_2$, which can be seen directly by using the corresponding formulas (1.12). If in (1.21) $T_{21}-T_{12}$ is replaced, in accordance with (1.18), by the expression $M_{21}R_2^{-1}-M_{12}R_1^{-1}$, and if we limit ourselves only to the main terms (this is equivalent to a replacement in (1.21) of γ_1/β_1, γ_2/β_2 by -1 and of the $M_{21}-M_{12}$-factor by -2; see above), then instead of the exact equality (1.21) we will obtain the approximate equality*

$$M_{21} - M_{12} \approx \frac{h^2}{24}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(T_{12} + T_{21}) \tag{1.22}$$

Thus the equalities (1.18) and (1.21) (or the approximate equality (1.22)) can be considered as two independent algebraic equations always relating T_{12}, T_{21}, M_{12}, M_{21}. It follows from (1.21) and (1.18), as well as from Formulas (1.12), that to each umbilic point of the shell's surface correspond equal values of the quantities M_{12}, M_{21} and T_{12}, T_{21} ($M_{12} = M_{21}$, $T_{12} = T_{21}$).

Thus, on the basis of the Kirchhoff-Love hypothesis, the components e_1, e_2, e_{12} of the shell deformation are determined by the quantities ϵ_1, ϵ_2, ω_1, ω_2, κ_1, κ_2, τ which are expressed in terms of the displacements u_1, u_2, u_3 in accordance with Formulas (1.3), (1.4), (1.8). Thereby, the forces and moments per unit length (except N_1 and N_2) are related with the quantities ϵ_1, ϵ_2, ..., τ by Formulas (1.12), while among themselves they are interrelated by five differential equations (1.15) to (1.17) and two algebraic equations (1.18) and (1.21) (or by the approximate equation (1.22)).

An important question arises regarding the possibility of simplifying the relations of elasticity (1.12). All existing versions of the theory of shells based on the Kirchhoff-Love hypothesis are dependent on the choice of certain approximate relations of elasticity. A rigorous solution of this question requires evaluation of relative discrepancies between the solutions (i.e. values of displacements and stresses) of an arbitrary boundary-value problem in the theory of shells based on the relations (1.12) and the simplified relations of elasticity. It is not clear how such an evaluation can be made for a shell of arbitrary shape. However, certain conclusions can be made referring to a cylindrical shell.

2. For the cylindrical shell the equalities (1.15) to (1.17) without loading terms, the relations of elasticity (1.12), and Formulas (1.3), (1.4), (1.18) yield a system of three differential equations

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* It differs from the "supplementary" equation derived in [1]. Equality (1.22) is fulfilled exactly, if in the corresponding formulas (1.12) only the main terms are considered, i.e. if it is assumed:

$$2(1+\nu)\,T_{i,3-i} = Eh\left[\omega - {}^1\!/_{12}\,(\chi_i - \chi_{3-i})\,h\tau\right]$$

$$24(1+\nu)\,M_{i,3-i} = -Eh^2\left[2h\tau + \chi_i\omega_{3-i} - (\chi_i - 2\chi_{3-i})\,\omega_i\right]$$

$$L_{j1}u_1 + L_{j2}u_2 + L_{j3}u_3 = 0 \quad (j=1,\,2,\,3) \tag{2.1}$$

where $L_{jk}(j,\ k = 1,\ 2,\ 3)$ are the following operators forming a symmetric matrix:

$$L_{11} = 2\frac{\partial^2}{\partial\xi^2} + (1-\nu)(1-\gamma)\frac{\partial^2}{\partial\varphi^2}, \qquad L_{12} = (1+\nu)\frac{\partial^2}{\partial\xi\,\partial\varphi}$$

$$L_{13} = -2\nu\frac{\partial}{\partial\xi} + 2\beta\frac{\partial^3}{\partial\xi^3} + (1-\nu)\gamma\frac{\partial^3}{\partial\xi\,\partial\varphi^2}, \quad L_{21} = (1+\nu)\frac{\partial^2}{\partial\xi\,\partial\varphi}$$

$$L_{22} = 2\frac{\partial^2}{\partial\varphi^2} + (1-\nu)(1+3\beta)\frac{\partial^2}{\partial\xi^2}, \qquad L_{23} = -2\frac{\partial}{\partial\varphi} + (3-\nu)\beta\frac{\partial^3}{\partial\xi^2\partial\varphi} \tag{2.2}$$

$$L_{31} = -2\nu\frac{\partial}{\partial\xi} + 2\beta\frac{\partial^3}{\partial\xi^3} + (1-\nu)\gamma\frac{\partial^3}{\partial\xi\,\partial\varphi^2}, \qquad L_{32} = -2\frac{\partial}{\partial\varphi} + (3-\nu)\beta\frac{\partial^3}{\partial\xi^2\,\partial\varphi}$$

$$L_{33} = 2(1-\gamma) - 4\gamma\frac{\partial^2}{\partial\varphi^2} + 2\beta\frac{\partial^4}{\partial\xi^4} + [(3+\nu)\beta - (1-\nu)\gamma]\frac{\partial^4}{\partial\xi^2\partial\varphi^2} - 2\gamma\frac{\partial^4}{\partial\varphi^4}$$

($R\xi$ and $R\phi$) are the coordinates in the axial and circumferential directions; $\beta = \beta_2,\ \gamma = \gamma_2$).

Let D be the determinant with the elements L_{jk}, and Λ_{jk} the corresponding minor of the L_{jk} element. The general solution* of the system (2.1) can be defined by the equations

$$u_1 = \Lambda_{31}\Phi, \qquad u_2 = -\Lambda_{32}\Phi, \qquad u_3 = \Lambda_{33}\Phi$$

where Φ is the general solution of equation $D\Phi = 0$. In expanded form this equation, reduced by the factor $4(1-\nu)\beta$ and in which the components $o(\beta)$ are neglected in the coefficients, becomes

$$\begin{aligned} L_1\Phi = {} & (1+3\beta)\frac{\partial^8\Phi}{\partial\xi^8} + \left(4 + \frac{11-3\nu}{2}\beta\right)\frac{\partial^8\Phi}{\partial\xi^6\,\partial\varphi^2} + 3[2 + (2-\nu)\beta]\frac{\partial^8\Phi}{\partial\xi^4\partial\varphi^4} + \\ & + \left(4 + \frac{7-3\nu}{2}\beta\right)\frac{\partial^8\Phi}{\partial\xi^2\partial\varphi^6} + (1+\beta)\frac{\partial^8\Phi}{\partial\varphi^8} + 2\nu(1+3\beta)\frac{\partial^6\Phi}{\partial\xi^6} + \\ & + 3[2 + (1-\nu)(2+\nu)\beta]\frac{\partial^6\Phi}{\partial\xi^4\,\partial\varphi^2} + [2(4-\nu) + (7-5\nu)\beta]\frac{\partial^6\Phi}{\partial\xi^2\,\partial\varphi^4} + \\ & + 2(1+\beta)\frac{\partial^6\Phi}{\partial\varphi^6} + \beta^{-1}[1-\nu^2 + (4-3\nu^2)\beta]\frac{\partial^4\Phi}{\partial\xi^4} + \\ & + \left[2(2-\nu) + \frac{7(1-\nu)}{2}\beta\right]\frac{\partial^4\Phi}{\partial\xi^2\,\partial\varphi^2} + (1+\beta)\frac{\partial^4\Phi}{\partial\varphi^4} = 0 \end{aligned} \tag{2.3}$$

* With the exception of pure elongation and torsion which are not considered here.

Without the terms $O(\beta)$ in the coefficients, the expression $L_1\Phi$ has the form

$$L_1\Phi = \Delta^4\Phi + 2\nu\frac{\partial^6\Phi}{\partial\xi^6} + 6\frac{\partial^6\Phi}{\partial\xi^4\,\partial\varphi^2} + 2(4-\nu)\frac{\partial^6\Phi}{\partial\xi^2\,\partial\varphi^4} + 2\frac{\partial^6\Phi}{\partial\varphi^6} +$$
$$+ (1-\nu^2)\,\beta^{-1}\frac{\partial^4\Phi}{\partial\xi^4} + 2(2-\nu)\frac{\partial^4\Phi}{\partial\xi^2\,\partial\varphi^2} + \frac{\partial^4\Phi}{\partial\varphi^4} \qquad (2.4)$$
$$(\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\varphi^2)$$

The same equation (2.3) is obtained if in the relations of elasticity (1.12), neglecting small quantities (see (1.13)), it is assumed that $\gamma_i/\beta_i = -1$ (which is equivalent to neglecting the quantities $O(\beta_i)$ and the last terms in the brackets in the expressions for M_i and $M_{i,3-i}$), i.e. if (1.12) are replaced by the following:

$$T_i = \frac{Eh}{1-\nu^2}\left[\varepsilon_i + \nu\varepsilon_{3-i} + \frac{1}{12}(\chi_i - \chi_{3-i})(\chi_i\varepsilon_i - h\varkappa_i)\right]$$
$$T_{i,\,3-i} = \frac{Eh}{2(1+\nu)}\left[\omega + \frac{1}{12}(\chi_i - \chi_{3-i})^2\,\omega_i - \frac{1}{12}(\chi_i - \chi_{3-i})\,h\tau\right]$$
$$M_i = -\frac{Eh^2}{12(1-\nu^2)}\left[h(\varkappa_i + \nu\varkappa_{3-i}) - (\chi_i - \chi_{3-i})\,\varepsilon_i\right] \qquad (2.5)$$
$$M_{i,\,3-i} = -\frac{Eh^2}{24(1+\nu)}\left[2h\tau + \chi_i\omega_{3-i} - (\chi_i - 2\chi_{3-i})\,\omega_i\right]$$

These formulas are essentially not different from the corresponding formulas derived in [3]. Formulas (2.5) for $T_{i,3-i}$ and $M_{i,3-i}$ can be written in the form

$$T_{i,\,3-i} = \frac{Eh}{2(1+\nu)}\left[\omega - \frac{1}{12}(\chi_i - \chi_{3-i})(h\tau^* - \chi_i\omega)\right]$$
$$M_{i,\,3-i} = -\frac{Eh^3}{24(1+\nu)}(2h\tau^* - \chi_i\omega)$$

In the presence of the relations (2.5), the exact equation (1.18) and the approximate equation (1.22) are valid. In place of (1.22) the following equation is fulfilled exactly:

$$\left[1 + \frac{h^2}{24}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2\right](M_{21} - M_{12}) = \frac{h^2}{24}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(T_{12} + T_{21})$$

(it is obtained from (1.21) if it is assumed that $\gamma_i/\beta_i = -1$). The change in the operators L_{jk} which occurs when the relations (1.12) are replaced by Formulas (2.5) consists merely in that the quantity γ is replaced by $-\beta$ (this does not disturb the symmetry of the matrix $\|L_{jk}\|$, i.e. some coefficients of the L_{jk} operators remain unchanged while others change by the amount $o(\beta)$, which is negligibly small compared to the values of these coefficients. The same happens to the operators by means of which (through the function Φ) are determined the displacements and the internal force factors, and consequently similar insignificant changes occur in the boundary conditions expressed by means of the function Φ. Taking all this into account, and based on the fact that the indicated small changes of the coefficients (of the order of $o(\beta)$) in the equation for Φ and in the boundary conditions expressed through Φ lead only to negligibly small changes of the function Φ itself, we conclude that for the cylindrical shell one can always replace the relations of elasticity (1.12) by the approximate relations (2.5) without essential error.

Further approximations of the relations (2.5) for the cylindrical shell can lead in particular cases to significant errors in the solution, as will be shown below.

3. Rejection of any (secondary) terms in Formulas (2.5) for $T_{i,3-i}$ and $M_{i,3-i}$ (here are considered the second and third terms in brackets of these formulas) leads to a violation of the sixth equation of equilibrium or to the appearance of stresses in the displacement of the shell as a rigid body. Naturally, these defects can lead to substantial errors, as was noted in [1]. Therefore, let us consider the more delicate question regarding the possibility for simplifying Formulas (2.5) for T_i and M_i. Let us refer to the following relations of elasticity, which differ from (2.5) only in the formulas for T_i and M_i:

$$T_i = \frac{Eh}{1-\nu^2}(\varepsilon_i + \nu\varepsilon_{3-i})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.1)$$
$$M_i = -\frac{Eh^3}{12(1-\nu^2)}(\varkappa_i + \nu\varkappa_{3-i})$$

If one accepts these relations, then in the equations (2.2), in addition to the replacement of γ by $-\beta$ (which differs from γ only by $o(\beta)$) there will occur a change in certain coefficients in a number of operators L_{jk} by the quantity $O(\beta)$ (such changes can substantially affect Φ; see p. 784).

Indeed, we will have

$$L_{13} = -2\nu\frac{\partial}{\partial\xi} - (1-\nu)\beta\frac{\partial^3}{\partial\xi\,\partial\varphi^2}, \quad L_{22} = 2(1+\beta)\frac{\partial^2}{\partial\varphi^2} + (1-\nu)(1+3\beta)\frac{\partial^2}{\partial\xi^2}$$
$$L_{23} = -2\frac{\partial}{\partial\varphi} + (3-\nu)\beta\frac{\partial^3}{\partial\xi^2\,\partial\varphi} + 2\beta\frac{\partial^3}{\partial\varphi^3}, \quad L_{31} = -2\nu\frac{\partial}{\partial\xi} - (1-\nu)\beta\frac{\partial^3}{\partial\xi\,\partial\varphi^2}$$
$$L_{32} = -2\frac{\partial}{\partial\varphi} + (3-\nu)\beta\frac{\partial^3}{\partial\xi^2\,\partial\varphi} + 2\beta\frac{\partial^3}{\partial\varphi^3}$$
$$L_{33} = 2 + 2\beta\frac{\partial^4}{\partial\xi^4} + 4\beta\frac{\partial^4}{\partial\xi^2\partial\varphi^2} + 2\beta\frac{\partial^4}{\partial\varphi^4}$$

(matrix $\| L_{jk} \|$ remains symmetric).

Then, instead of Equations (2.3), (2.6) one obtains the following:

$$L_2\Phi = (1+3\beta)\frac{\partial^8\Phi}{\partial\xi^8} + \left[4+\frac{3}{2}(1+\nu)\beta\right]\frac{\partial^8\Phi}{\partial\xi^6\partial\varphi^2} + [6+3(1-\nu)\beta]\frac{\partial^8\Phi}{\partial\xi^4\partial\varphi^4} +$$
$$+\left[4+\frac{3}{2}(1-\nu)\beta\right]\frac{\partial^8\Phi}{\partial\xi^2\partial\varphi^6} + (1+\beta)\frac{\partial^8\Phi}{\partial\varphi^8} + [6-3\nu(1-\nu)\beta]\frac{\partial^6\Phi}{\partial\xi^4\partial\varphi^2} +$$
$$+[8+3(1-\nu)\beta]\frac{\partial^6\Phi}{\partial\xi^2\partial\varphi^4} + 2(1+\beta)\frac{\partial^6\Phi}{\partial\varphi^6} + \beta^{-1}(1-\nu^2)(1+3\beta)\frac{\partial^4\Phi}{\partial\xi^4} +$$
$$+\left[4+\frac{3}{2}(1-\nu)\beta\right]\frac{\partial^4\Phi}{\partial\xi^2\partial\varphi^2} + (1+\beta)\frac{\partial^4\Phi}{\partial\varphi^4} = 0 \qquad (3.2)$$

$$L_2\Phi = \Delta^4\Phi + 6\frac{\partial^6\Phi}{\partial\xi^4\partial\varphi^2} + 8\frac{\partial^6\Phi}{\partial\xi^2\partial\varphi^4} + 2\frac{\partial^6\Phi}{\partial\varphi^6} +$$
$$+(1-\nu^2)\beta^{-1}\frac{\partial^4\Phi}{\partial\xi^4} + 4\frac{\partial^4\Phi}{\partial\xi^2\partial\varphi^2} + \frac{\partial^4\Phi}{\partial\varphi^4} \qquad (3.3)$$

In the case when the shell is acted upon by a normal pressure $q = q(\xi, \phi)$ we will have the following, instead of Equations (2.3), (3.2):

$$L_1\Phi = 6(1+\nu)R^4q / h^3E \qquad (3.4)$$
$$L_2\Phi = 6(1+\nu)R^4q / h^3E \qquad (3.5)$$

If one chooses q and the boundary conditions for the shell such that in the solution of Equation (3.4), corresponding to this choice of the boundary conditions, the term with the large parameter β^{-1} would vanish, it would then be natural to expect that a significant difference in certain terms containing sixth and fourth derivatives in Equations (3.4), (3.5) (see also (2.4), (3.2)) will substantially affect their solution.

In view of this, let us assume

$$q = 3q_0(3\xi^2 - 6 - \nu)\cos 2\varphi \qquad (3.6)$$

Then the particular solution of Equation (3.4) will be

$$\Phi = \frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\left[\frac{\xi^2}{1+\beta} - \frac{1+3\nu}{4}\frac{\beta}{(1+\beta)^2}\right]\cos 2\varphi \approx$$
$$\approx \frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\xi^2\cos 2\varphi - \frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\beta\left(\xi^2 + \frac{1+3\nu}{4}\right)\cos 2\varphi \qquad (3.7)$$

(the small term with the β-multiplier on the right-hand side of Equation (3.7) is attributed to the small terms in the L_1 operator coefficients from (2.3); the reason for the retention of these small quantities is explained in the footnote*).

The following displacements, forces and moments per unit length correspond to the particular solution (3.7):

$$u_1 = 6(1-\nu^2)\frac{R^4 q_0}{h^3 E}\xi\cos 2\varphi, \qquad u_2 = 6(1-\nu^2)\frac{R^4 q_0}{h^3 E}\left(\xi^2 - 1 - \frac{\nu}{2}\right)\sin 2\varphi$$

$$u_3 = 12(1-\nu^2)\frac{R^4 q_0}{h^3 E}(\xi^2 - 1)\cos 2\varphi \tag{3.8}$$

$$T_1 = \left[6(1-\nu^2)\frac{R^2}{h^2} + \frac{1}{2}(1-\nu)(7+\nu)\right] R q_0 \cos 2\varphi$$

$$T_{12} = -3(1-\nu) R q_0 \xi \sin 2\varphi, \quad M_1 = -R^2 q_0 [{}^5/_2 - \nu(3\xi^2 - 3 + {}^1/_2\nu)]\cos 2\varphi$$

$$M_{12} = 3(1-\nu) R^2 q_0 \xi \sin 2\varphi, \qquad N_1 = \frac{1}{R}\left(\frac{\partial M_1}{\partial \xi} + \frac{\partial M_{21}}{\partial \varphi}\right) = 6 R q_0 \xi \cos 2\varphi$$

$$T_2 = R q_0 (3\xi^2 - 6 + \nu)\cos 2\varphi, \qquad T_{21} = 0 \tag{3.9}$$

$$M_2 = -R^2 q_0 (3 + 2\nu - 3\xi^2)\cos 2\varphi, \qquad M_{21} = 3(1-\nu) R^2 q_0 \xi \sin 2\varphi$$

$$N_2 = \frac{1}{R}\left(\frac{\partial M_2}{\partial \varphi} + \frac{\partial M_{12}}{\partial \xi}\right) = R q_0 (9 + \nu - 6\xi^2)\sin 2\varphi$$

Formulas (3.8), (3.9), with the exception of T_1, contain no small terms which, compared to the included terms, are as small as h^2/R^2 is small compared to unity. But in the derivation of these formulas the small terms were considered when the main terms were mutually eliminated.

The small term in T_1 is retained because it is of the same order of magnitude as the remaining forces.

At the boundaries of the shell (for $\xi = \pm\rho = \pm l/2R$, l is the length of the shell) we obtain from (3.9)

* If in the definitions for ω and $\epsilon_2 + \nu\epsilon_1$ one considers only the usual main terms in the operators Λ_{jk}, and only the main terms in Formula (3.7), we will then obtain $\omega = 0$, $\epsilon_2 + \nu\epsilon_1 = 0$. Retention of the small term in Formula (3.7) significantly affects the value of T_2. Equation (3.9) for T_2 is obtained with this term included. Even though T_2 appears negligibly small compared to T_1, its change prevents fulfilment of the equations of equilibrium. Therefore, in order to avoid possible difficulties, the small quantities are retained in Equations (2.3), (3.2), (3.7) and below in the equation for Φ_1.

$$T_1 = [6(1-\nu^2)R^2/h^2 + {}^1/_2(1-\nu)(7+\nu)]Rq_0\cos 2\varphi$$
$$T_{12} - M_{12}/R = \mp 6(1-\nu)Rq_0\rho\sin 2\varphi$$
$$M_1 = -R^2q_0[{}^5/_2 - \nu(3\rho^2 - 3 + {}^1/_2\nu)]\cos 2\varphi \qquad (3.10)$$
$$N_1 + (1/R)(\partial M_{12}/\partial\varphi) = \pm 12(1-\nu)Rq_0\rho\cos 2\varphi$$

Consequently, assuming the elasticity relations (2.5), then for the cylindrical shell subjected to the pressure (3.6) and with the boundary conditions (3.10), we obtain the displacements (3.8) and the forces and moments per unit length (3.9). Thereby, the state of stress of the shell is practically determined by the force T_1, i.e. the stresses corresponding to the forces T_{12}, N_1, T_2, T_{21}, N_2 and the moments M_1, M_{12}, M_2, M_{21} are negligibly small compared to the stress corresponding to T_1. This stress is $\sigma_1 = Eh(\epsilon_1 + \nu\epsilon_2)/(1-\nu^2) \approx T_1/h$.

In passing we will show how for an arbitrary shell the stresses σ_1, σ_2, σ_{12} can be expressed in terms of the forces T_i, $T_{i,3-i}$ and the moments M_i, $M_{i,3-i}$, if these forces and moments are determined from (2.5). From the first formula (1.10) and the first and third relation (2.5) (if ϵ_i, κ_i are expressed in terms of T_i, M_i by means of these relations) for $z = \pm h/2$ we have

$$\sigma_i = \frac{T_i}{h} \pm \frac{6M_i}{h^2} + \frac{T_i}{h}[o(\chi_i) + O(\chi_{3-i})] + \frac{T_{3-i}}{h}[o(\chi_{3-i}) + O(\chi_i\chi_{3-i})] +$$
$$+ \frac{M_i}{h^2}[O(\chi_i) + O(\chi_{3-i})] + \frac{M_{3-i}}{h^2}[O(\chi_i) + O(\chi_{3-i})]$$

Therefore, if T_i and M_i are determined according to (2.5), then these stresses can be computed by the usual formula

$$\sigma_i = T_i/h \pm 6M_i/h^2$$

with a negligibly small error as compared with the largest stress σ_1, σ_2 (for fixed a_1, a_2 and $z = \pm h/2$).

Furthermore, inasmuch as $z/R_i \ll 1$, Formula (1.10) can be replaced by the approximate equation

$$\sigma_{12} = \frac{E}{2(1+\nu)}\left[\omega - 2z\tau + \left(\frac{z}{R_1} - \frac{z}{R_2}\right)(\omega_1 - \omega_2)\right]$$

(the last term in brackets is retained for the case if $\omega = 0$, i.e. $\omega_1 = -\omega_2$). Hence, for $z = \pm h/2$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\left\{\omega \pm \frac{1}{2}[(\chi_1 - \chi_2)(\omega_1 - \omega_2) - 2h\tau]\right\}$$

From this equation and

$$T_{12}+T_{21}=\frac{Eh}{1+\nu}\left[1+\frac{1}{24}(\chi_1-\chi_2)^2\right]\omega\approx\frac{Eh}{1+\nu}\,\omega$$

$$\chi_1 M_{12}-\chi_2 M_{21}=\frac{Eh^2}{24(1+\nu)}(\chi_1-\chi_2)\left[(\chi_1-\chi_2)(\omega_1-\omega_2)-2h\tau\right]$$

resulting from the second and the fourth formula in (2.5), it follows that for $z = \pm h/2$

$$\sigma_{12}=\frac{T_{12}+T_{21}}{2h}\pm\frac{6}{h^2}\,\frac{M_{12}R_1^{-1}-M_{21}R_2^{-1}}{R_1^{-1}-R_2^{-1}}$$

(for $T_{12} = T_{21}$, $M_{12} = M_{21}$ this formula becomes the ordinary formula for σ_{12}).

Let us turn now to the relations (3.1) and Equation (3.5), where q is determined from (3.6). The particular solution of Equation (3.5) is

$$\Phi_1=\frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\left\{\frac{\xi^2}{1+\beta}-\left[\frac{\nu}{3}+\frac{5}{12}(3+\nu)\beta\right]\frac{1}{(1+\beta)^2}\right\}\cos 2\varphi\approx$$
$$\approx\frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\left(\xi^2-\frac{\nu}{3}\right)\cos 2\varphi-\frac{3}{8}(1+\nu)\frac{R^4q_0}{h^3E}\beta\left(\xi^2+\frac{5-\nu}{4}\right)\cos 2\varphi$$

The following quantities correspond to the function Φ_1:

$$u_1=6(1-\nu^2)\frac{R^4q_0}{h^3E}\xi\cos 2\varphi,\quad u_2=6(1-\nu^2)\frac{R^4q_0}{h^3E}\left(\xi^2-1-\frac{5}{6}\nu\right)\sin 2\varphi$$
$$u_3=12(1-\nu^2)\frac{R^4q_0}{h^3E}\left(\xi^2-1-\frac{\nu}{3}\right)\cos 2\varphi \qquad (3.11)$$
$$T_1=\left[6(1-\nu^2)\frac{R^2}{h^2}+\frac{1}{2}(1-\nu)(7+\nu)+\nu(3\xi^2-\nu)\right]Rq_0\cos 2\varphi$$
$$T_{12}=-3Rq_0\xi\sin 2\varphi$$
$$M_1=-R^2q_0\left[2-\nu\left(3\xi^2-3-\frac{\nu}{2}\right)\cos 2\varphi,\quad M_{12}=3(1-\nu)R^2q_0\xi\sin 2\varphi\right.$$
$$N_1=6Rq_0\xi\cos 2\varphi \qquad (3.1)$$
$$T_2=(3\xi^2-6-\nu)Rq_0\cos 2\varphi,\qquad T_{21}=-3\nu\xi Rq_0\sin 2\varphi$$
$$M_2=-R^2q_0(3+{}^5/_2\nu-3\xi^2)\cos 2\varphi,\qquad M_{21}=3(1-\nu)R^2q_0\xi\sin 2\varphi$$
$$N_2=(9+2\nu-6\xi^2)Rq_0\sin 2\varphi$$

In (3.11) and (3.12) there are no terms of the same order of magnitude as in (3.8), (3.9). Quantities u_1, M_{12}, N_1, T_2, M_{21} and the main part of T_1 are unchanged. For $\xi = \pm\rho$ we will again have the fourth equation (3.10), while the remaining equations (3.10) are replaced by the following:

$$T_1 = \left[6(1-\nu^2)\frac{R^2}{h^2} + \frac{1}{2}(1-\nu)(7+\nu) + \nu(3\rho^2-\nu)\right] Rq_0 \cos 2\varphi$$

$$T_{12} - \frac{M_{12}}{R} = \mp 3(2-\nu) Rq_0\rho \sin 2\varphi, \quad M_1 = -R^2 q_0 \left[2 - \nu\left(3\rho^2 - 3 - \frac{\nu}{2}\right)\right] \cos 2\varphi$$

In order to obtain a solution of Equation (3.5) which corresponds to T_1, T_{12}, N_1, M_1, M_{12} according to Formulas (3.1), satisfying all conditions of (3.1), we add to Φ_1 a correspondingly chosen solution Φ_0 of the homogeneous equation (3.2). The function Φ_0 can be expressed in the form

$$\Phi_0 = (A_1 \cosh k\xi \cos k\xi + B_1 \sinh k\xi \sin k\xi + A_2 \cosh k_1\xi \cos k_2\xi + B_2 \sinh k_1\xi \sin k_2\xi) \cos 2\varphi \qquad (3.13)$$

Here*

$$k \approx \varkappa, \quad k_1 \approx (\sqrt{3}/\varkappa)(1 + 3/2\varkappa^2), \quad k_2 \approx (\sqrt{3}/\varkappa)(1 - 3/2\varkappa^2)$$
$$(\varkappa^4 = 3(1-\nu^2) R^2/h^2)$$

and the constants A_1, A_2, B_1, B_2 must be chosen such that T_1, T_{12}, N_1, M_1, M_{12} (quantities corresponding to Φ_0 which will be denoted by T_1^0, T_{12}^0, ...) satisfy the conditions

$$T_1^\circ = \nu(\nu - 3\rho^2) Rq_0 \cos 2\varphi, \qquad M_1^\circ = R^2 q_0 (\nu^2 - 1/2) \cos 2\varphi$$
$$T_{12}^\circ - M_{12}^\circ / R = \pm 3\nu Rq_0\rho \sin 2\varphi, \quad N_1^\circ + (1/R)\,\partial M_{12}^\circ / \partial\varphi = 0 \qquad (3.14)$$

for $\xi = \pm\rho$.

Neglecting quantities of the order of h/R as compared to unity one can, for example, write the equations for T_1^0 and M_1^0 as

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* Small terms by which differ k_1 and k_2 are retained because they significantly affect some derivatives of cosh $k_1\xi$ cos $k_2\xi$ and cosh $k_1\xi$ sin $k_2\xi$ used in forming Equation (3.15). For example $(\cosh k_1\xi \cos k_2\xi)' = k_1 \sinh k_1\xi \cos k_2\xi - k_2 \cosh k_1\xi \sin k_2\xi \approx 6\kappa^{-4}\xi(3-\xi^2)$ while, if it is assumed that $k_1 = k_2 = k_0 = \sqrt{3}/K$, then $(\cosh k_0\xi \cos k_0\xi)' = k_0(\sinh k_0\xi \cos k_0\xi - \cosh k_0\xi \sin k_0\xi) \approx -6\kappa^{-4}\xi^3$.

$$T_1^\circ = -2(1-\nu)\frac{Eh}{R}\frac{\partial^4}{\partial\xi^2\partial\varphi^2}[(A_1\cosh k\xi\cos k\xi + B_1\sinh k\xi\sin k\xi + A_2\cosh k_1\xi\cos k_2\xi + B_2\sinh k_1\xi\sin k_2\xi)\cos 2\varphi]$$

$$M_1^\circ = -\frac{Eh^3}{6(1+\nu)R^2}\left\{\frac{\partial^6}{\partial\xi^6}[(A_1\cosh k\xi\cos k\xi + B_1\sinh k\xi\sin k\xi)\cos 2\varphi] + \nu\left(\frac{\partial^6}{\partial\varphi^6}+\frac{\partial^4}{\partial\varphi^4}\right)(A_2 \quad k_1\xi\cos k_2\xi\cos 2\varphi) + \left[\nu\left(\frac{\partial^6}{\partial\varphi^6}+\frac{\partial^4}{\partial\varphi^4}\right) + (1+2\nu)\frac{\partial^6}{\partial\xi^2\partial\varphi^4} + \nu(2+\nu)\frac{\partial^4}{\partial\xi^2\partial\varphi^2}\right](B_2\sinh k_1\xi\sin k_2\xi\cos 2\varphi)\right\}$$

and similarly simple expressions for T_{12}^0, N_1^0, M_{12}^0. On the basis of these expressions conditions (3.14) become

$$\begin{aligned} b_1A_1 - a_1B_1 + a_2A_2 - b_2B_2 &= c_1\varkappa^{-6} \\ b_1A_1 - a_1B_1 - a_2'A_2 - b_2'B_2 &= c_2\varkappa^{-6} \\ b_1'A_1 - a_1'B_1 + a_3A_2 - b_3B_2 &= c_3\varkappa^{-7} \\ b_1'A_1 - a_1'B_1 - a_3'A_2 - b_3'B_2 &= 0 \end{aligned} \qquad (3.15)$$

where*

$$a_1 = \cosh k\rho\cos k\rho, \qquad a_1' = \sinh k\rho\cos k\rho - \cosh k\rho\sin k\rho$$
$$b_1 = \sinh k\rho\sin k\rho, \qquad b_1' = \cosh k\rho\sin k\rho + \sinh k\rho\cos k\rho$$
$$a_2 \approx 9\varkappa^{-6}(\rho^2-1), \qquad b_2 \approx 3\varkappa^{-4}, \qquad a_2' \approx 6\nu\varkappa^{-6}$$
$$b_2' \approx 6\nu\varkappa^{-8}(3\rho^2 - 3 + {}^1/_2\nu - 2/\nu), \qquad a_3 \approx 18\varkappa^{-7}\rho$$
$$b_3 \approx 18\varkappa^{-9}\rho(6-\rho^2+1-{}^1/_2\nu), \qquad a_3' \approx 18\varkappa^{-11}\rho[2(2-\nu)(3-\rho^2)+17-8\nu]$$
$$b_3' \approx 36(2-\nu)\varkappa^{-9}\rho, \qquad c_1 = {}^3/_{16}\nu(1+\nu)(3\rho^2-\nu)(R^4q_0/h^3E)$$
$$c_2 = {}^3/_4(1+\nu)({}^1/_2-\nu^2)(R^4q_0/h^3E), \qquad c_3 = {}^9/_8\nu(1+\nu)\rho(R^4q_0/h^3E)$$

From (3.15), assuming $\kappa^{-1} \ll \rho \ll \kappa$ and neglecting quantities of the order of h/R as compared to unity, we obtain

$$\begin{aligned} A_1 &= -\frac{3}{4}\frac{q_0R^4}{Eh^3}(1+\nu)(1-\nu^2)\varkappa^{-6}\frac{\sinh\varkappa\rho\cos\varkappa\rho - \cosh\varkappa\rho\sin\varkappa\rho}{\sinh 2\varkappa\rho + \sin 2\varkappa\rho} \\ B_1 &= -\frac{3}{4}\frac{q_0R^4}{Eh^3}(1+\nu)(1-\nu^2)\varkappa^{-6}\frac{\cosh\varkappa\rho\sin\varkappa\rho + \sinh\varkappa\rho\cos\varkappa\rho}{\sinh 2\varkappa\rho + \sin 2\varkappa\rho} \\ A_2 &= \frac{1}{16}\frac{q_0R^4}{Eh^3}\nu(1+\nu), \qquad B_2 = \frac{1}{8}\frac{q_0R^4}{Eh^3}\varkappa^{-2}(1+\nu)\left(1-\frac{3}{2}\nu-\frac{\nu^2}{2}\right) \end{aligned} \qquad (3.16)$$

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* The indicated coefficients in A_2, B_2 were obtained with the aid of such approximate equalities as

$$a_2 = {}^1/_2\varkappa^{-2}[2k_1k_2\sinh k_1\rho\sin k_2\rho - (k_1^2-k_2^2)\cosh k_1\rho\cos k_2\rho] \approx 9\varkappa^{-6}(\rho^2-1)$$

Let us note concurrently the following. On the basis of (3.13), (3.16)

$$\Phi_0 \approx A_2 \cosh k_1 \xi \cos k_2 \xi \cos 2\varphi \approx \frac{1}{16} \frac{q_0 R^4}{Eh^3} \nu (1+\nu) \cos 2\varphi$$

Consequently, the main part of the function $\Phi_0 + \Phi_1$ which is equal to

$$\frac{3}{8}(1+\nu)\frac{R^4 q_0}{h^3 E}\left(\xi^2 - \frac{\nu}{6}\right)\cos 2\varphi$$

differs from the main part of the function Φ defined by Formula (3.7). Thus it is established that the change in the coefficients of some operators Λ_{jk} by the quantity $O(\beta)$ (which corresponds to the passage from (2.5) to (3.1)) can substantially alter the solving function Φ, corresponding to the full solution of the boundary-value problem. Utilizing (3.16) and neglecting quantities of the order of h/R as compared to unity, we find

$$u_1^\circ = -3\nu(1-\nu^2)\frac{q_0 R^4}{Eh^3}\varkappa^{-3}\left[(1-\nu^2) f_2(\xi) + \left(\xi^2 + \frac{5}{2}\nu - \frac{2}{\nu}\right)\xi\varkappa^{-1}\right]\cos 2\varphi$$

$$u_2^\circ = \nu(1-\nu^2)\frac{q_0 R^4}{Eh^3}\cosh\frac{\sqrt{3}}{\varkappa}\xi\cos\frac{\sqrt{3}}{\varkappa}\xi\sin 2\varphi \approx \nu(1-\nu^2)\frac{q_0 R^4}{Eh^3}\sin 2\varphi \qquad (3.17)$$

$$u_3^\circ = 2\nu(1-\nu^2)\frac{q_0 R^4}{Eh^3}\cosh\frac{\sqrt{3}}{\varkappa}\xi\cos\frac{\sqrt{3}}{\varkappa}\xi\cos 2\varphi \approx 2\nu(1-\nu^2)\frac{q_0 R^4}{Eh^3}\cos 2\varphi$$

(u_1^0, u_2^0, u_3^0 are displacements corresponding to Φ_0)

$$\begin{aligned}
T_1^\circ &= q_0 R\{2(1-\nu^2)[1+2f_1(\xi)] + \nu(\nu - 3\xi^2)\}\cos 2\varphi \\
T_2^\circ &= q_0 R[{}^1/_2\nu - 2(1-\nu^2)\varkappa^2 f_2(\xi)]\cos 2\varphi \\
T_{12}^\circ \approx T_{21}^\circ &= q_0 R[4(1-\nu^2)\varkappa f_3(\xi) + 3\nu\xi]\sin 2\varphi \qquad (3.18)\\
N_1^\circ &= -2(1-\nu^2) q_0 R\varkappa f_3(\xi)\cos 2\varphi \\
N_2^\circ &= -2q_0 R[(1-\nu^2) f_1(\xi) + {}^1/_2\nu]\sin 2\varphi \\
M_1^\circ &= q_0 R^2[(1-\nu^2) f_1(\xi) + {}^1/_2\nu^2]\cos 2\varphi \\
M_2^\circ &= q_0 R^2\nu[(1-\nu^2) f_1(\xi) + {}^1/_2]\cos 2\varphi \\
M_{12}^\circ \approx M_{21}^\circ &= 2(1-\nu)(1-\nu^2) q_0 R^2\varkappa^{-1} f_4(\xi)\sin 2\varphi
\end{aligned}$$

Here

$$f_1(\xi) = [(\sinh\varkappa\rho\cos\varkappa\rho - \cosh\varkappa\rho\sin\varkappa\rho)\sinh\varkappa\xi\sin\varkappa\xi - (\cosh\varkappa\rho\sin\varkappa\rho + \sinh\varkappa\rho\cos\varkappa\rho)\cosh\varkappa\xi\cos\varkappa\xi](\sinh 2\varkappa\rho + \sin 2\varkappa\rho)^{-1}$$

$$f_2(\xi) = [(\sinh\varkappa\rho\cos\varkappa\rho - \cosh\varkappa\rho\sin\varkappa\rho)\cosh\varkappa\xi\cos\varkappa\xi - (\cosh\varkappa\rho\sin\varkappa\rho + \sinh\varkappa\rho\cos\varkappa\rho)\sinh\varkappa\xi\sin\varkappa\xi](\sinh 2\varkappa\rho + \sin 2\varkappa\rho)^{-1}$$

$$f_3(\xi) = (\cosh\varkappa\rho\sin\varkappa\rho\sinh\varkappa\xi\cos\varkappa\xi - \sinh\varkappa\rho\cos\varkappa\rho\cosh\varkappa\xi\sin\varkappa\xi)(\sinh 2\varkappa\rho + \sin 2\varkappa\rho)^{-1}$$

$$f_4(\xi) = (\cosh\varkappa\rho\sin\varkappa\rho\cosh\varkappa\xi\sin\varkappa\xi + \sinh\varkappa\rho\cos\varkappa\rho\sinh\varkappa\xi\cos\varkappa\xi)(\sinh 2\varkappa\rho + \sin 2\varkappa\rho)^{-1}$$

Taking into account the limitation $\kappa^{-1} \ll \rho \ll \kappa$ and keeping in mind that $|\xi| \leqslant \rho$, it is easy to obtain the following evaluations:

$$|f_1(\xi)| < 3, \qquad |f_2(\xi)| < 3, \qquad |f_3(\xi)| < 1.5, \qquad |f_4(\xi)| < 1.5$$

The quantities $u_1{}^0$ and $T_1{}^0$ are therefore negligibly small as compared to $u_1{}^1$ and $T_1{}^1$ which correspond to the solution Φ_1. Consequently

$$u_1 = 6(1-\nu^2)\frac{R^4 q_0}{h^3 E}\xi\cos 2\varphi, \qquad u_2 = 6(1-\nu^2)\frac{R^4 q_0}{h^3 E}\left(\xi^2 - 1 - \frac{2}{3}\nu\right)\sin 2\varphi$$
$$u_3 = 12(1-\nu^2)\frac{R^4 q_0}{h^3 E}\left(\xi^2 - 1 - \frac{1}{6}\nu\right)\cos 2\varphi, \quad T_1 = 6(1-\nu^2)\frac{R^3 q_0}{h^2}\cos 2\varphi \tag{3.19}$$

will correspond to the solution $\Phi_0 + \Phi_1$.

Also, from (3.18) and (3.12) it follows that the state of stress corresponding to the solution $\Phi_0 + \Phi_1$ is practically determined by the force T_1.

Thus, based on the relations of elasticity (3.1), the cylindrical shell, subjected to the pressure (3.6) with the boundary conditions (3.10), obtains practically the same state of stress as that based on the relation (2.5). Likewise, the same displacement u_1 is obtained, but the displacements u_2, u_3, which are of the same order as u_1, are different. For $\rho \leqslant \sqrt{2}$ ($l \leqslant 2\sqrt{2}\,R$) $\nu = 0.3$, the difference in the maximum absolute values of u_3 (i.e. $|u_3|$ when $\xi = 0$) is 5 per cent, as can be seen from Formulas (3.8), (3.9). Although this discrepancy is not large, it is important that it does not depend on h/R in this regard it is significant. It is understood that the indicated discrepancy is not connected with the displacement of the shell as a rigid body, since such a displacement is defined by the quantities u_1, u_2, u_3 of the form

$$u_1 = a + a'\cos\varphi + a''\sin\varphi, \qquad u_2 = b + (b' + a'\xi)\sin\varphi + (b'' - a''\xi)\cos\varphi$$
$$u_3 = (b' + a'\xi)\cos\varphi - (b'' - a''\xi)\sin\varphi$$

(a, a', a'', b, b', b'' are arbitrary constants). The fact that the difference between the two derived values of displacements u_2, u_3 (see (3.8), (3.19)) is proportional to ν is due to the fact that the difference between the corresponding coefficients in Equations (3.4), (3.5) is also proportional to ν (see (2.4), (3.3)).

Analogous results are obtained if instead of (3.1) one utilizes the relations of elasticity which differ from (2.5) only by the formula $T_i = [Eh/(1-\nu^2)]\,(\epsilon_i + \nu\epsilon_{3-i})$ or the formula $M_i = -[Eh^3/12(1-\nu^2)] \times (\kappa_i + \nu\kappa_{3-i})$. As far as the simplification of Formula (2.5) for T_i by means of eliminating one of the quantities $\chi_i\epsilon_i$ or $h\kappa_i$ is concerned, in the general case it is without foundation: these quantities, generally speaking, are of the same order of magnitude.*

4. The example presented is not unique. Similar examples occur for any loading of the type $q = Q(\xi) \cos n\phi$, where $n = 2, 3, \ldots,$ and $Q(\xi)$ a polynomial of second or third power. Then Equations (3.4), (3.5) have particular solutions (of the same form as q) for which the term with the large factor β^{-1} vanishes in Expressions (2.4), (3.3), while the terms which distinguish Expressions (2.4), (3.3) from one another are not negligibly small as compared to the other terms in these expressions if n is relatively small. Such solutions, apparently, differ substantially (in the above-indicated sense) from one another (with the increase in n the difference decreases, therefore $n = 2$ in the case considered). The presence of these particular solutions of Equations (3.4), (3.5) for certain boundary conditions leads to significantly different functions Φ, which causes a substantial difference in the corresponding displacements.

In these examples the differences in maximum stresses became negligibly small (relative difference is of the order of $O(h/R)$). Examples of a different kind can be given when the elasticity relations (2.5), (3.1) formally lead to substantially different maximum stresses. This difference occurs not because of the differences in the solving function, but for another reason. The reason is that for certain discontinuous loadings, in the formulas defining the internal force factors by means of the solving function, the terms possessing a singularity depend on the choice of the elasticity relations, particularly the relations for bending moments.

Let us turn to the case when a tangential force Q_1 (in the axial direction) or Q_2 (in the circumferential direction), uniformly distributed on a rectangular element σ, acts on a cylindrical shell surface. The element σ is bounded by two segments of generators with the length $2a = 2R\alpha$ and two arcs of lengths $2b = 2R\beta$. In this case, of all internal force factors, only $N_2 = N_2^{(1)}$ is unbounded (for Q_1 acting) or $N_1 = N_1^{(2)}$ (for Q_2 acting). They are unbounded in the neighborhood of the corner points (with the coordinates $\xi = \pm\alpha$, $\phi = \pm\beta$) of the loaded element σ, and to them correspond the maximum stresses (this does not contradict the Kirchhoff-Love hypothesis in the above-presented formulation). On the basis of the elasticity relations coinciding in regard to T_i, M_i with (3.1), the following asymptotic formulas were obtained in [5] for $N_2^{(1)}$ and $N_1^{(2)}$:

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* One should note that for the cylindrical shell the discarding of $\chi_i\epsilon_i$ in Formula (2.5) for T_i does not alter Equation (2.3).

$$N_2^{(1)} \approx \mp \frac{1+\nu}{64\pi} \frac{h^2 Q_1}{Rs} \ln \rho, \qquad N_1^{(2)} \approx \mp \frac{7-\nu}{192\pi} \frac{h^2 Q_2}{Rs} \ln \rho$$

$$\rho^2 = (\xi \pm \alpha)^2 + (\varphi \pm \beta)^2$$

Here $s = 4ab$ is the surface of the element σ, while h is the thickness of the shell which in [5] is denoted by $2h$. In utilizing the elasticity relations (2.5) one obtains other, more harmonious formulas

$$N_2^{(1)} \approx \mp \frac{1+\nu}{48\pi} \frac{h^2 Q_1}{Rs} \ln \rho, \qquad N_1^{(2)} \approx \mp \frac{1-\nu}{48\pi} \frac{h^2 Q_2}{Rs} \ln \rho.$$

They differ from the preceding* ones by 33.3 per cent and 58.2 per cent respectively, independently of the value of h/R.

In this case, one can justify the simplified relations (3.1) only on practical grounds, pointing out the following circumstances. If a and b are commensurate with R, then the $N_2^{(1)}$ and $N_2^{(2)}$ forces determined from the asymptotic formulas are, respectively, smaller (in magnitude) than

$$(h^2/R^2)(Q_1/R)\,|\ln \rho| \text{ и } (h^2/R^2)(Q_2/R)\,|\ln \rho|$$

while the forces $T_1^{(1)}$ and $T_2^{(2)}$, roughly speaking, are characterized by the quantities Q_1/R and Q_2/R. Therefore, the forces $N_2^{(1)}$ and $N_1^{(2)}$ can exceed $T_1^{(1)}$ and $T_2^{(2)}$, respectively, only at points which are extremely near the corners, at a distance of $r = R\rho < h(R/h) \exp(-R^2/h^2) \ll h$.

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* From the elasticity relations used in [5], the forces $N_2^{(1)}$ and $N_1^{(2)}$ are expressed by means of the solving function with the aid of the operators containing, respectively, the following combinations of eighth-order derivatives (from the various derivatives entering into these operators only the eighth derivatives possess singularities):

$$(1+\nu)\frac{\partial^8}{\partial\xi^7\partial\varphi} + (1+\nu)\frac{\partial^8}{\partial\xi^5\partial\varphi^3}, \qquad 2\frac{\partial^8}{\partial\xi^7\partial\varphi} + (3-\nu)\frac{\partial^8}{\partial\xi^5\partial\varphi^3} + (1-\nu)\frac{\partial^8}{\partial\xi^3\partial\varphi^5}$$

Instead of these expressions one obtains the following ones by using the relations (2.5):

$$(2+\nu)\frac{\partial^8}{\partial\xi^7\partial\varphi} + (3+2\nu)\frac{\partial^8}{\partial\xi^5\partial\varphi^3} + \nu\frac{\partial^8}{\partial\xi^3\partial\varphi^5} - \frac{\partial^8}{\partial\xi\,\partial\varphi^7}$$

$$2\frac{\partial^8}{\partial\xi^7\partial\varphi} + (3-\nu)\frac{\partial^8}{\partial\xi^5\partial\varphi^3} - 2\nu\frac{\partial^8}{\partial\xi^3\partial\varphi^5} - (1+\nu)\frac{\partial^8}{\partial\xi\,\partial\varphi^7}$$

This then leads to the indicated difference in the asymptotic formulas.

In practice, however, instead of having corner points, the boundary of the σ-element will be rounded with radii which are larger than the given values of r. Also larger than these values of r will be the width of a zone adjacent to the boundary of σ where the actual loading will be attenuated. Consequently, in this case, the existence of points at which $N_2^{(1)}$ and $N_2^{(2)}$ are primary is unrealistic. If, however, a and b are sufficiently small, then the element σ can be considered practically as an oval-shaped region with a varying radius $r = R\rho$, and it may be assumed that the largest value of the forces $T_1^{(1)}$, $N_2^{(1)}$ ($T_2^{(2)}$, $N_1^{(2)}$) will approximately equal the values of these forces on the boundary of the σ-element, due to the action of the force $Q_1(Q_2)$ located at the center of σ. They can be found from the asymptotic formulas derived in [5, pp. 169, 170], from which it follows that $N_2^{(1)}(N_1^{(2)})$ can be larger than $T_1^{(1)}$ ($T_2^{(2)}$) only for $r \ll h$, which is also unrealistic.

Nevertheless, in the theoretical sense, one cannot neglect to consider the difference in the derived asymptotic formulas.

5. In connection with the question considered, one should mention [6]. In it, basically, the following conclusion was drawn: based on the Kirchhoff-Love hypothesis, the elasticity relations should be assumed in the simplest form, i.e. they should be expressed as for the plate. This conclusion is based on the fact that the hypotheses, more general than the Kirchhoff-Love hypothesis, lead to the formula for M_1 in particular, which differs from the corresponding formula (2.5) by three additional terms, one or maybe two of which (the evaluation of the second term in [6] is not sufficiently substantiated) are of the same order of magnitude as the term ϵ_1 in the referred formula (2.5). Strictly speaking, nothing yet follows from this, since the algebraic sum can be a quantity of higher order of smallness than the individual terms. Also, even if the sum of the three indicated additional terms is a quantity of the same order as the first one of them, it is not impossible that under even more general hypotheses or by using the equations of the theory of elasticity, the correction to Formula (2.5) for M_1 (at least for a particular class of problems) will be a quantity of higher order of smallness than the ϵ_1-term in this formula. Finally, the various corrections in the elasticity relations, being of the same order of magnitude, can affect the solution by different amounts (see footnote on p. 786). Therefore, there is no sufficient basis for asserting that the error introduced by the Kirchhoff-Love hypothesis is a quantity of the same order of magnitude as the corrections obtained by means of improvement in the simplest relations of elasticity, even if only the relations for T_i and M_i are considered. As far as such corrections are concerned, they can be substantial, as has been established. Note, by the way, that the effects developed here from the simplification of Formulas (2.5) for T_i, M_i are also obtained by simplification of Formulas (2.5) for

$T_{i,3-i}$, $M_{i,3-i}$.

Thus the consideration presented in [1], the results of [2] and the investigation carried out in the present paper lead to the following conclusion: if, in addition to the error of the Kirchhoff-Love hypothesis, one admits only an error of the order of h/R, then, in the general case (for an arbitrary shell and loading), it is unlawful to replace the relations of elasticity (1.12) by simpler formulas than (2.5).

6. In conclusion, we note a curious fact, in view of which the utilized example of the cylindrical shell, subjected to the pressure (3.6) and with boundary conditions (3.10), becomes of independent interest.

Let us set $q_0 = q_* h^2/R^2$ in Equation (3.6), where $q_* = 1$ kg/cm^2. Assuming $\xi \leqslant \rho < \sqrt{(R/h)}$ we find that for sufficiently small h/R the quantity max $|q|$ is arbitrarily small. But, meanwhile, there will be acting in the shell a rigorously constant (non-attenuated) along the length, axial force T_i, self-equilibrated at each end of the shell, and equal to $6Rq_*$ $\cos 2\phi$ according to Formula (3.9) (this quantity remains constant as max $|q|$ decreases with a decrease in h/R). The stress corresponding to this force, $\sigma_1 = 6Rh^{-1}q_* \cos 2\phi$ will be arbitrarily large inasmuch as h/R is regarded sufficiently small. For example, for $\rho = 3$, $(1 = 6R)$ and $h/R = 1/400$ we will have max $|q| = 0.000376$ kg/cm^2, i.e. the pressure is practically absent. But due to this negligible pressure, the force $T_1 = 6Rq_* \cos 2\phi$, which is self-equilibrated at each end of the shell, remains rigorously constant along the whole length of the shell, whereby max $|\sigma_1| = 2400$ kg/cm^2. Should the pressure q be rigorously equal to zero, then on the strength of the Saint-Venant principle, the end-equilibrated force $T_1 \neq 0$ could not remain constant along the length of the shell. The example presented is qualitatively different from the known case when the surface loading is rigorously equal to zero, while the self-equilibrated load T_1 at each end of the shell remains constant only practically along the length of the shell because of quite slow attenuation (such a case can be realized if one should take for Φ Expression (3.13) with $A_1 = B_1 = A_2 = 0$ and assume static boundary conditions which are satisfied for the given choice of Φ).

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